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AFFINE CATEGORIES AND NATURALLY MAL'CEV CATEGORIES

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This note is a complement to a beautiful recent paper of A. Carboni, which characterizes affine categories, i.e. those categories which occur as slices of additive categories. We show that the condition that every reflexive graph has a unique groupoid structure, which was observed by Carboni to follow from affineness, is equivalent to the existence of a natural Mal'cev operation on a category; we further show that this condition implies the additiveness of the category of pointed objects, but not the affineness of the original category.

This note is a complement to a beautiful recent paper of Carboni [1], which characterizes affine categories (i.e. those categories which occur as slices of (finitely complete) additive categories) by means of a 'modularity' condition relating products and coproducts and a 'positivity' condition relating coproducts and pullbacks. At the end of [1], Carboni notes that an affine category $\mathscr{C}$ also has the property that the forgetful functor from internal groupoids to reflexive graphs in $\mathscr{C}$ is an isomorphism, and raises the question (which he attributes to F.W. Lawvere) whether this condition is itself sufficient to characterize affine categories. In this paper we shall show that the answer is no: more precisely, we shall show that the above property of reflexive graphs is equivalent to the existence of a natural Mal'cev operation on $\mathscr{C}$, that it implies the additiveness of the category of pointed objects of $\mathscr{C}$, and that neither this implication nor the implication "affine $\Rightarrow$ naturally Mal'cev" is reversible.

We recall (cf. [3]) that a *Mal'cev operation* on a set A is a function $p: A^3 \to A$ satisfying the equations

$$p(x,y,y)=x \quad \text{and} \quad p(x,x,y)=y \tag{1}$$

for all $x,y\in A$. We shall say a Mal'cev operation p is *associative* if it also satisfies

$$p(x,y,p(u,v,w))=p(p(x,y,u),v,w), \tag{2}$$

commutative if it satisfies $p(x,y,z)=p(z,y,x)$, and *autonomous* (cf. [2]) if it com-

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mutes with itself, i.e.

$$\begin{aligned} &p(p(x_1,x_2,x_3),p(y_1,y_2,y_3),p(z_1,z_2,z_3)) \\ &\quad = p(p(x_1,y_1,z_1),p(x_2,y_2,z_2),p(x_3,y_3,z_3)) \end{aligned} \tag{3}$$

holds for all values of the variables involved. An autonomous Mal'cev operation is associative and commutative, as may be seen by simplifying the two sides of the equations

$$p(p(x,x,x),p(y,x,x),p(u,v,w)) = p(p(x,y,u),p(x,x,v),p(x,x,w))$$

and

$$p(p(x,x,x),p(x,x,y),p(z,y,y)) = p(p(x,x,z),p(x,x,y),p(x,y,y)).$$

Further, we have

Lemma. *If p and q are two Mal'cev operations on the same set A which commute with each other, then they coincide (and are autonomous).*

Proof. Simplify the two sides of the equation

$$p(q(x,x,x),q(y,x,x),q(y,y,z)) = q(p(x,y,y),p(x,x,y),p(x,x,z))$$

which is a particular case of the assertion that p and q commute. $\square$

In particular, a commutative (= autonomous [2]) algebraic theory can contain at most one Mal'cev operation.

The above arguments all belong to universal algebra, and so can be reproduced in the internal logic of any category with finite products. By a *natural Mal'cev operation* on a category $\mathscr{C}$ with finite products, we mean a natural transformation p from the functor $A \mapsto A^3$ to the identity functor on $\mathscr{C}$ such that p_A is a Mal'cev operation on A, for each A; we say $\mathscr{C}$ is a *naturally Mal'cev category* if it admits a natural Mal'cev operation. The naturality of p with respect to $p_A : A^3 \to A$ (and the three product projections) tells us immediately that a natural Mal'cev operation is autonomous; further, the lemma tells us that for any object A of a naturally Mal'cev category p_A is the unique Mal'cev operation on A – so that in particular a given category with finite products admits at most one natural Mal'cev operation.

If $\mathscr{C}$ is a category with finite limits, we shall write Refl($\mathscr{C}$), Cat($\mathscr{C}$) and Gpd($\mathscr{C}$) respectively for the categories of internal reflexive graphs in $\mathscr{C}$ (i.e. diagrams

$$\begin{tikzcd} A_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & A_0 \arrow[l, bend left=60, "s"] \end{tikzcd}$$

satisfying $d_0 s = d_1 s = 1_{A_0}$), internal categories in $\mathscr{C}$ and internal groupoids in $\mathscr{C}$.

We have forgetful functors

$$\mathrm{Gpd}(\mathscr{C}) \to \mathrm{Cat}(\mathscr{C}) \to \mathrm{Refl}(\mathscr{C})$$

of which the first is always full, but the second is not in general. We may now state our main result:

Theorem. *For a category $\mathscr{C}$ with finite limits, the following are equivalent:*
(i) *$\mathscr{C}$ is naturally Mal'cev.*
(ii) *The forgetful functor* $\mathrm{Gpd}(\mathscr{C}) \to \mathrm{Refl}(\mathscr{C})$ *is an isomorphism.*
(iii) *The forgetful functor* $\mathrm{Cat}(\mathscr{C}) \to \mathrm{Refl}(\mathscr{C})$ *is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii). Let $\mathbf{A} = (A_0, A_1, d_0, d_1, s)$ be a reflexive graph in $\mathscr{C}$, and define A_2 by the pullback square

$$\begin{array}{ccc} A_2 & \xrightarrow{d_0} & A_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} \\ A_1 & \xrightarrow{d_0} & A_0 \end{array}$$

We must define morphisms $d_1 : A_2 \to A_1$ (composition) and $i : A_1 \to A_1$ (inversion) satisfying the appropriate equations to make $\mathbf{A}$ into an internal groupoid. We take d_1 to be the composite

$$A_2 \xrightarrow{(d_0, sd_1d_0, d_2)} A_1^3 \xrightarrow{p_{A_1}} A_1$$

and i to be the composite

$$A_1 \xrightarrow{(sd_1, 1_{A_1}, sd_0)} A_1^3 \xrightarrow{p_{A_1}} A_1.$$

The verification that this is indeed a groupoid structure is straightforward: for example, the 'book-keeping' axioms $d_0d_1 = d_0d_0$ and $d_1d_1 = d_1d_2$ follow from the naturality of p, and the associativity of composition follows from the associativity (in the sense of (2)) of p_{A_1} – we have already observed that this follows from its autonomy. Moreover, the naturality of p ensures that the operation of endowing A with this structure is a functor $\mathrm{Refl}(\mathscr{C}) \to \mathrm{Gpd}(\mathscr{C})$, left inverse to the forgetful functor. So, to complete the proof of (ii), we need only verify that this is the unique possible groupoid structure on $\mathbf{A}$.

Let $m : A_2 \to A_1$ be any composition making $\mathbf{A}$ into an internal category. By naturality of p, we have a commutative diagram

$$\begin{array}{ccccc} A_2 & \xrightarrow{(s_1d_2, s_0sd_1d_0, s_0d_0)} & A_2^3 & \xrightarrow{p_{A_2}} & A_2 \\ & \searrow{\scriptstyle (d_0, sd_1d_0, d_2)} & {\scriptstyle m^3}\downarrow & & \downarrow{\scriptstyle m} \\ & & A_1^3 & \xrightarrow{p_{A_1}} & A_1 \end{array}$$

where $s_0 : A_1 \to A_2$ is defined by $d_0 s_0 = 1_{A_1}$ and $d_2 s_0 = s d_1$, and s_1 by $d_0 s_1 = s d_0$ and $d_2 s_1 = 1_{A_1}$. So, to show that m is the composite d_1 defined earlier, it suffices to show that the top composite (f, say) in the above diagram is the identity on A_2. But we also have a commutative diagram

$$\begin{tikzcd}
A_2 \arrow[r, "{(s_1 d_2, s_0 s d_1 d_0, s_0 d_0)}"] \arrow[dr, "{(s d_1 d_0, s d_1 d_0, d_0)}"'] & A_2^3 \arrow[r, "p_{A_2}"] \arrow[d, "d_0^3"] & A_2 \arrow[d, "d_0"] \\
& A_1^3 \arrow[r, "p_{A_1}"] & A_1
\end{tikzcd}$$

and the bottom composite here is d_0 because p_{A_1} is a Mal'cev operation; thus $d_0 f = d_0$, and similarly $d_2 f = d_2$. Since A_2 is a pullback, this implies $f = 1_{A_2}$ and hence $m = d_1$. The uniqueness of the inverse map $A_1 \to A_1$ is similarly established.

(iii) ⇒ (i). Given an object A of $\mathscr{C}$, let $T(A)$ be the reflexive graph

$$\begin{tikzcd}
A^3 \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_3"'] & A \arrow[l, bend left=60, "\Delta_3"]
\end{tikzcd}$$

where π_1 and π_3 are the first and third projections and Δ_3 is the diagonal map. T is clearly a functor $\mathscr{C} \to \mathrm{Refl}(\mathscr{C})$; so by (iii) it lifts to a functor $\mathscr{C} \to \mathrm{Cat}(\mathscr{C})$; i.e. we can impose a natural category structure on $T(A)$. The 'object of composable pairs' of $T(A)$ may be identified with A^5; we now define p_A to be the composite

$$A^3 \xrightarrow{\Delta_2 \times 1_A \times \Delta_2} A^5 \xrightarrow{\quad d_1 \quad} A^3 \xrightarrow{\quad \pi_2 \quad} A$$

where d_1 is the composition map of the category $T(A)$, and Δ_2 is the diagonal map $A \to A^2$. The naturality of p follows from the naturality of the category structure on $T(A)$; the fact that it is a Mal'cev operation from (the diagrams expressing) the fact that composition with an identity morphism is the identity operation in a category.

(ii) ⇒ (iii). (ii) clearly implies that the forgetful functor $\mathrm{Cat}(\mathscr{C}) \to \mathrm{Refl}(\mathscr{C})$ has a left inverse, which is all we used in proving (iii) ⇒ (i); so (ii) implies (i). But the proof of (i) ⇒ (ii) showed that, if (i) holds, every reflexive graph in $\mathscr{C}$ has a unique category structure, so (iii) holds. □

Remark. It is possible for $\mathrm{Cat}(\mathscr{C})$ and $\mathrm{Gpd}(\mathscr{C})$ to coincide without the conditions of the theorem being satisfied. For example, it is well known that this coincidence occurs when $\mathscr{C}$ is the category **Gp** of not-necessarily-abelian groups, but **Gp** is not naturally Mal'cev. (The forgetful functor $\mathbf{Gp} \to \mathbf{Set}$ admits a Mal'cev operation, but the identity functor $\mathbf{Gp} \to \mathbf{Gp}$ does not.)

Let $\mathscr{C}$ be a naturally Mal'cev category. If B is an object such that pullbacks over B exist in $\mathscr{C}$, then the slice category $\mathscr{C}/B$ is naturally Mal'cev; given an object $(f: A \to B)$ of $\mathscr{C}/B$, we define its Mal'cev operation to be the restriction of p_A to the fibre product $A \times_B A \times_B A \rightarrowtail A^3$ (and note that this is a morphism over B, because the restriction of p_B to the diagonal $B \rightarrowtail B^3$ is the identity on B). Similarly, the co-slice category $B \setminus \mathscr{C}$ is naturally Mal'cev for any object B of $\mathscr{C}$; in particular, taking B to be the terminal object, the category $\mathrm{Pt}(\mathscr{C})$ of pointed objects of $\mathscr{C}$ is naturally Mal'cev. However, we have

Proposition. *For a category $\mathscr{C}$ with finite products, the following are equivalent:*
(i) *$\mathscr{C}$ is additive.*
(ii) *$\mathscr{C}$ is naturally Mal'cev and has a zero object.*

Proof. In an additive category, we may define the natural Mal'cev operation by the formula $p(x, y, z) = x - y + z$ (interpreted in the internal logic of $\mathscr{C}$); conversely, given a natural Mal'cev operation p and a zero 0, we may define addition by $x + y = p(x, 0, y)$ and negation by $-x = p(0, x, 0)$. The fact that addition is associative and commutative follows from the associativity and commutativity of p; the fact that 0 is an identity for addition, and $-x$ an inverse for x, follows from the fact that p is a Mal'cev operation. □

Corollary. *If $\mathscr{C}$ is a naturally Mal'cev category, then the category* $\mathrm{Pt}(\mathscr{C})$ *is additive.*

Proof. Combine the proposition with the remarks before it. □

We conclude with a couple of counterexamples. Let $\mathscr{C}$ be the opposite of the category **Set** (more generally, we could take the opposite of any non-degenerate topos); then $\mathrm{Pt}(\mathscr{C})$, being the opposite of **Set**/0, is degenerate and hence additive. But it is easy to see that there cannot be a 'Mal'cev co-operation' (i.e. a function $A \to A + A + A$ satisfying the duals of the Mal'cev conditions (1)) on any nonempty set A, let alone a natural Mal'cev operation on $\mathbf{Set}^{\mathrm{op}}$. So the corollary does not provide a characterization of naturally Mal'cev categories.

On the other hand, the presence of a natural Mal'cev operation does not characterize affine categories. This is easily seen by considering any non-degenerate meet-semilattice as a category; for a less trivial example, take the category $\mathbb{T}$-Alg, where $\mathbb{T}$ is the algebraic theory generated by a ternary operation p satisfying (1) and (3). Since $\mathbb{T}$ is by construction a commutative theory, $\mathbb{T}$-Alg is naturally Mal'cev; moreover, if we add a constant to the theory $\mathbb{T}$ we obtain the theory of abelian groups, by the argument of the proposition, so that $\mathrm{Pt}(\mathbb{T}\text{-Alg})$ is isomorphic to the category **Ab** of abelian groups. However, $\mathbb{T}$-Alg is not equivalent to a slice of **Ab**, or of any additive category (for example, it has a strict initial object, which is impossible for a slice of a non-degenerate additive category). It is worth noting that $\mathbb{T}$-Alg does satisfy the 'positivity' condition (condition (2) of the definition of modularity) of [1]; i.e. for every morphism $f: A \to B$ and every object C, the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A+C \arrow[r, "f+1"] & B+C \end{tikzcd}$$

is a pullback (and the vertical arrows in it are monomorphisms). So the conjunction of this condition with the possession of a natural Mal'cev operation is not sufficient to imply modularity.

References

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